Paul Hindemith & Howard Hanson
Two Different Approaches, One Same Goal.

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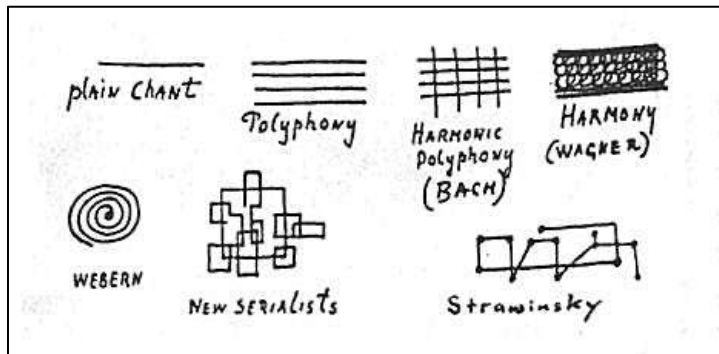# Introduction

Stravinsky himself made the drawing below. It is a fun, yet interesting, representation of his view of the evolution of music. His sketches show the increasing complexity of the music, especially when studying the last hundred and fifty years.

Theory has not always adjusted fast and good enough along the way, and its methods several times seem inadequate for the more contemporary works. After the extreme chromaticism of the late Romantic period, the return to a form of modalism, the advent of alternatives to tonality at the beginning of the XXth century, then the coexistence of many different trends (sometimes radically different), the revival of Classical forms but with the influence of modern times, etc., it looked like necessary for Paul Hindemith to come up with a more comprehensive text to help students to master the vast harmonic vocabulary at their disposal. He published *The Craft of Musical Composition* in 1937.

Some twenty years later, Howard Hanson, a college professor with extensive experience with composition students concluded that it was necessary a basic text that could give the young composer the necessary guidance for studying the material of the art and for showing how it could be used. Thus, he published *Harmonic Materials of Modern Music* in 1960.

A generation apart, both composers/educators have very different approaches but yet, one same goal: to master the harmonic materials. First themselves, and then providing a path for their students and future composers. This work will highlight the main points of both books will compare some aspects of them. Then, a piece by a third composer will be analyzed with insights from both theories.

Paul Hindemith

The Medium

After describing in the introduction of his book, the historical problems that have often led to what he calls "confusion in the technique of composition", interspersing his own experience as both a composer and a teacher, Paul Hindemith focuses in the second part of the book on the nature of the sound. He explains general concepts related to the overtone series, how each frequency is calculated in this series, the first attempts to form scales and the necessity to sacrifice purity in order to have polyphony: the tempered tuning.

Paul Hindemith

Hindemith addresses the problem of the Pythagorean comma and proposes a new method to determine the frequencies of the overtone series more accurately, including the problem of the seventh overtone. For the purpose of this paper, I will not go deeper into the first part of his book, focusing myself on the following part.

Series 1 and 2

Given a tone, Hindemith sustains that an octave higher stands in so close a relationship that it is difficult to have a distinction between the two. After this interval, the perfect fifth higher than the given tone is the most closely related, then follow the perfect fourth, the major sixth, the major third, the minor third, the minor sixth, the major second, the minor seventh, the minor second and the major seventh. This is what he called *series 1*. As the distance in this series increases, the relationship with the first tone diminishes. He does not include the tritone, though he says that at the distance of an augmented fourth or diminished fifth, the relationship between the two tones is almost inexistent.

series 1
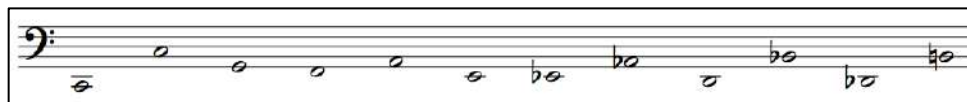
The values of the relationships, established in this series will determine the connection of tones and chords, the harmonic progressions, and hence, the tonal progress of a composition. Just as tone-relations are arranged in a descending order of value[i] in series 1, so are the harmonic intervals, which conform the *series 2*. To construct this order, Hindemith introduces the concept of *combination tones*. When two instruments of the same kind or a single stringed instrument plays a double stop, additional tones –in the low register– are involuntarily produced and they blend as part of the general timbre. While overtones are produced by a single sounding tone, combin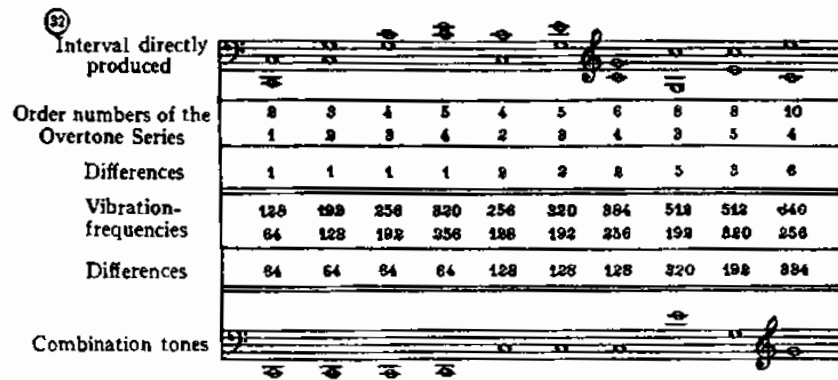ation tones appear when two or more tones sound together. This phenomenon, though known since early times, in tuning forks or in the construction of organs, had never been applied in music theory until Hindemith. The following figure shows the combination tones produced by different intervals.
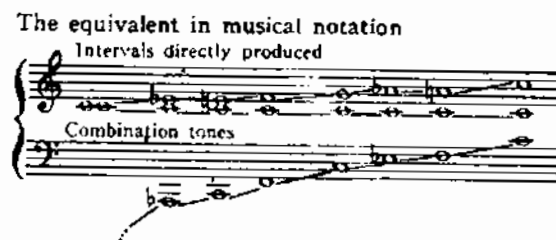
It can be seen that, for example in the case a major third C-E, an additional tone C two octaves lower is produced. But if the interval is a perfect fifth C-G, the combination tone is also a C but now only one octave lower. If the given interval is an unison, a perfect fifth or fourth, or a major third, the combination tone will be one of the same class of one component of the given interval. The octave is a particular case, because the combination tone produced coincides in pitch class and also in register with the lowest member of the octave. In all these cases, there is a reinforcement of the interval sound but the new tone belongs to the same pitch class of at least one of the notes of the interval. However, if the given interval is a minor third, or a sixth, the combination tone is a new pitch class.

The following table shows the different frequencies at which appear the combination tones.

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[i] Although Hindemith talks about values, he does not provide exact values for the relationships between the given tone and the tones above it at the different intervals.

Order numbers of the Overtone Series									
2	3	4	5	4	5	6	6	8	10
1	2	3	4	2	3	4	3	5	4

Differences									
1	1	1	1	2	2	2	5	3	6

Vibration-frequencies									
128	192	256	320	256	320	384	512	512	640
64	128	192	256	128	192	256	192	320	256

Differences									
64	64	64	64	128	128	128	320	192	384

As a general rule, the frequency of the combination tone is always the difference between the frequencies of the two tones that conform the interval. Moreover, if it is considered the difference in frequency between the given interval and the combination tone produced, combination tones of second order appear, following a curve that goes in the opposite direction to the first order combination tones.

The level of a combination tone is never loud enough to mask the directly produced interval. However, it gives the interval a characteristic *color*. Except for the unison and the octave that do not present combination tones different from themselves, starting with the fifth the different intervals have a varying weight of the combination tones. The perfect fourth seems to be affected a little more than the perfect fifth.

The major third and the minor sixth show two doublings of one of the constituent tones in their combination tones. In the major third the lower tone is doubled, in the minor sixth, the upper. Both have, in turn, one tone that is not contained in the directly produced interval. The minor third and major sixth have two of these new tones. Taking into account then the

combination tones of first and second orders, and knowing that the deeper tones, with slower vibration rates, have a greater weight than higher ones, it is possible to conclude that the fifth is superior to the fourth. Likewise, the major third is superior to the minor sixth, the major second to the minor seventh, etc. This way, a new series called *series 2*, summarizes the different harmonic value of the intervals:

Interval root

The tone that is more strongly influenced by the combination tones is called the root of the interval, being the other tone of the secondary importance. Thus, in the P5, the M3 and the m3 the root is the lower tone, whereas in the P4, the m6 and the M6, the root is the upper one. In seconds and sevenths, it is a little different. In seconds, M or m, the root is the upper tone, while in sevenths, M or m, the root is the lower one.

The tritone does not have a root and goes at the end of *series 2*.

Harmony and melody

Harmony and melody complement each other. Every interval has melodic and harmonic characteristics. They are distributed according to the following figure:

While intervals such as the fifth or the major third have stronger harmonic value, seconds and sixths are essentially melodic because of the attraction that other tones exert upon them, like the octave or the fifth. The tritone has no significance, either melodic or harmonic. It needs a third tone; if this third tone sounds simultaneously with the tritone, then the latter has a harmonic significance; if the tritone is part of a three-note melody, then it will have a more melodic function.

A new system of chords

The traditional theory of harmony offered to Hindemith several limitations. On one hand, the basic principle to construct chords was the superposition of thirds. This limited the possible combinations to a very small group of sonorities. And to explain notes out of these stack of thirds, theory has to recur sometimes to strange explanations. On the other hand, all chords must be invertible, which is difficult even with ninth chords, which are part of the possible ones. Besides, any tones out of the diatonic scale were considered alterations or distortions of the system, while other departures –like the lowered sixth degree or the Neapolitan sixth chord– were accepted with no complain. Finally, chords were susceptible of many reinterpretations.

Hindemith came up with a new system in which the third was no longer the only basis for constructing chords, the invertibility of chords obeys a more general principle, and chords are not susceptible to various interpretations. On both ends of the spectrum of series 2, are the octave and the tritone. The former has no significance for chord analysis and the tritone is so characteristic that every chord containing a tritone acquires some indefiniteness. So the first criteria to classify chords will be to separate in two groups: one with chords that contain no tritones (group A), and another with chords containing one or more tritones (group B). Then, chords will differentiate according to the presence or absence of the intervals that are more dissonant: seconds and sevenths. Thus, a subdivision can be made between those chords with no tritone and with or without sevenths and seconds (A1 and A2); and chords with tritone but with or without seventh and seconds (B1 and B2). Further differentiation has to do with the position of the root of the chord. It has been explained how to find the root of an interval. The concept of a chord root is explained below.

When a chord is in root position, root and bass coincide. The root, being already the strongest tone of the chords, it is reinforced by being the lowest tone. When the root is in another position, the two forces separate; the root opposes the strength of the bass. So chords whose bass and root

are not the same note, are subordinate to the version of those chords that do have both root and bass in the same place. Hindemith also states that the rearrangement of a chord may affect its quality. While chords with mild dissonances are not greatly affected by a more open disposition of its tones, a more dissonant chord may lose all its quality by disposing its tones more separate from each other.

Chord root

Except for a few exceptions, every chord contains a root. To find it, it is necessary to follow series 2. The root of the most important interval (most to the left in the series 2) present in the chord will be, in turn, the root of the whole chord. Thus, if the chord contains a perfect fifth, the lower tone of that interval is the root of the chord. If the chord presents a major third as the most important interval, its lower tone is the root of the chord. If no present interval is more important than a major second, then the upper tone of this interval is the root of the chord. If two intervals of most importance are present, the one that is lower in register has more preponderance over the other, and its root is the root of the whole chord.

The register is not important to determine the root. It does not matter if the tone that completes the interval that determines the root is in the same octave or lies two octaves higher (for fifths, thirds and sevenths) or lower (for fourths, sixths and seconds).

Guide-tone

When dealing with chords containing tritones, the root is calculated as usual, but in addition one of the notes that form the tritone serves as the guide-tone. The guide tone is:

1) the note of the tritone that is in best relationship with the root, as measured in series 2;
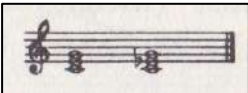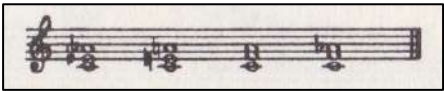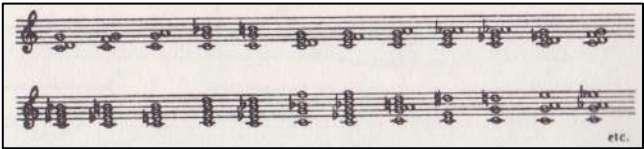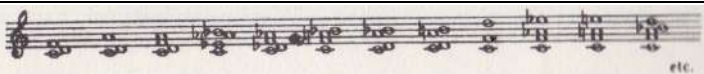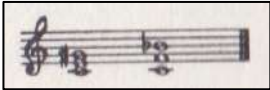
2) if the root forms part of the tritone, then the other note of the tritone is the guide-tone.
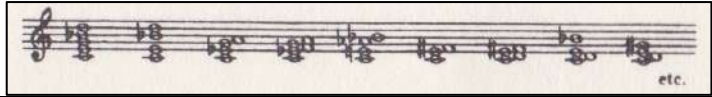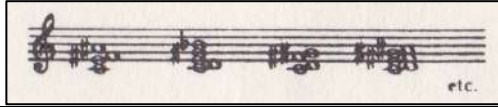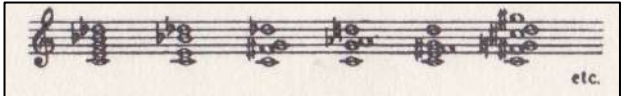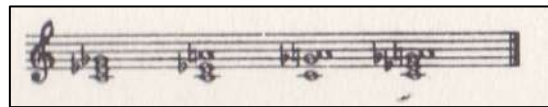
Table of Chord-Groups

The following is the chart of chords proposed by Hindemith:

A. Chords without tritone	
I. Without seconds or sevenths	1. Root and base tones are identical
	2. Root lies above the base tone
III. Containing seconds or sevenths or both	1. Root and base tone are identical
	2. Root lies above the base tone
V. Indeterminate	

			B. Chords containing tritone	
	a. With minor seventh only. Root and bass are identical.			
II. Without minor seconds or major sevenths. The tritone subordinate.	**b.** Containing major seconds or minor sevenths or both.	1. Root and base tones are identical		
		2. Root lies above the base tone		
		3. Containing more than one tritone		
IV. With minor seconds or major sevenths, or both. One or more tritones subordinate.	1. Root and base tone are identical			
	2. Root lies above the base tone			
VI. Indeterminate. Tritone predominating.				

Observations in Hindemith's table of chords[ii]

With no tritone

a) Groups I and V only have trichords.

b) For chords with no tritone, tetrads and larger groups only belong to group III.

With tritone

c) Group VI only has two members: the diminished triad and the fully diminished chord.

d) There are only two pentads that belong to group II, (02468) and (02469). All the other pentads belong to group IV.

e) There is only one hexad that belong to group II, (02468t). All the other hexads belong to group IV.

f) All heptads, octads and nonads belong to group IV.

Pitch Class Set	A (no tritone)			B (with tritone)					Forte
	I 1/2	III 1/2	V	II.a	II.b 1/2	II.b 3	IV 1/2	VI	
012									3 1
013									3 2
014									3 3
015		III							3 4
016									3 5
024									3 6
025		III							3 7
026				II.a	II.b12				3 8
027									3 9
036								VI	3 10
037	I								3 11
048			V						3 12

Pitch Class Set	A (no tritone)			B (with tritone)					Forte
	I 1/2	III 1/2	V	II.a	II.b 1/2	II.b 3	IV 1/2	VI	
0123		III							4 1
0124		III							4 2
0134		III							4 3
0125		III							4 4
0126							IV		4 5
0127							IV		4 6
0145		III							4 7
0156							IV		4 8
0167							IV		4 9
0235		III							4 10
0135		III							4 11
0236							IV		4 12
0136							IV		4 13
0237		III							4 14
0146							IV		4 15
0157							IV		4 16
0347		III							4 17
0147							IV		4 18
0148		III							4 19
0158		III							4 20
0246					II.b12				4 21
0247		III							4 22
0257		III	V						4 23
0248					II.b12				4 24
0268						II.b3			4 25
0358		III							4 26
0258				II.a					4 27
0369								VI	4 28
0137							IV		4 29

ii A complete table with Hindemith, Hanson and Forte combined appears in the appendix.

Paul Hindemith — "The Craft of Musical Composition / Theory" (1942)

Five-note sets

Pitch Class Set	A (no tritone) I 1/2	A (no tritone) III 1/2	V	B II a	B II b 1/2	B II b 3	B IV 1/2	VI	
01234		III							5 1
01235		III							5 2
01245		III							5 3
01236							IV		5 4
01237							IV		5 5
01256							IV		5 6
01267							IV		5 7
02346							IV		5 8
01246							IV		5 9
01346							IV		5 10
02347		III							5 11
01356							IV		5 12
01248							IV		5 13
01257							IV		5 14
01268							IV		5 15
01347							IV		5 16
01348		III							5 17
01457							IV		5 18
01367							IV		5 19
01378							IV		5 20
01458		III							5 21
01478							IV		5 22
02357		III							5 23
01357							IV		5 24
02358							IV		5 25
02458							IV		5 26
01358		III							5 27
02368							IV		5 28
01368							IV		5 29
01468							IV		5 30
01369							IV		5 31
01469							IV		5 32
02468						II.b3			5 33
02469					II.b12				5 34
02479		III							5 35
01247							IV		5 36
03458		III							5 37
01258							IV		5 38

Six-note sets

Pitch Class Set	A (no tritone) I 1/2	A (no tritone) III 1/2	V	B II a	B II b 1/2	B II b 3	B IV 1/2	VI	
012345		III							6 1
012346							IV		6 2
012356							IV		6 3
012456							IV		6 4
012367							IV		6 5
012567							IV		6 6
012678							IV		6 7
023457		III							6 8
012357							IV		6 9
013457							IV		6 10
012457							IV		6 11
012467							IV		6 12
013467							IV		6 13
013458		III							6 14
012458							IV		6 15
014568							IV		6 16
012478							IV		6 17
012578							IV		6 18
013478							IV		6 19
014589		III							6 20
023468							IV		6 21
012468							IV		6 22
023568							IV		6 23
013468							IV		6 24
013568							IV		6 25
013578							IV		6 26
013469							IV		6 27
013569							IV		6 28
013689							IV		6 29
013679							IV		6 30
013589							IV		6 31
024579		III							6 32
023579							IV		6 33
013579							IV		6 34
02468T						II.b3			6 35
012347							IV		6 36
012348							IV		6 37
012378							IV		6 38
023458							IV		6 39
012358							IV		6 40
012368							IV		6 41
012369							IV		6 42
012568							IV		6 43
012569							IV		6 44
023469							IV		6 45
012469							IV		6 46
012479							IV		6 47
012579							IV		6 48
013479							IV		6 49
014679							IV		6 49

Seven-note sets

Pitch Class Set	A (no tritone) I 1/2	A (no tritone) III 1/2	V	B II a	B II b 1/2	B II b 3	B IV 1/2	VI	
0123456							IV		7 1
0123457							IV		7 2
0123458							IV		7 3
0123467							IV		7 4
0123567							IV		7 5
0123478							IV		7 6
0123678							IV		7 7
0234568							IV		7 8
0123468							IV		7 9
0123469							IV		7 10
0134568							IV		7 11
0123479							IV		7 12
0124568							IV		7 13
0123578							IV		7 14
0124678							IV		7 15
0123569							IV		7 16
0124569							IV		7 17
0123589							IV		7 18
0123679							IV		7 19
0124789							IV		7 20
0124589							IV		7 21
0125689							IV		7 22
0234579							IV		7 23
0123579							IV		7 24
0234679							IV		7 25
0134579							IV		7 26
0124579							IV		7 27
0135679							IV		7 28
0124679							IV		7 29
0124689							IV		7 30
0134679							IV		7 31
0134689							IV		7 32
012468T							IV		7 33
013468T							IV		7 34
013568T							IV		7 35
0123568							IV		7 36
0134578							IV		7 37
0124578							IV		7 38

Pitch Class Set	A (no tritone)			B (with tritone)						
	I (1/2)	III (1/2)	V	II a	II b 1/2	II b 3	IV (1/2)	VI		
01234567							IV		8	1
01234568							IV		8	2
01234569							IV		8	3
01234578							IV		8	4
01234678							IV		8	5
01235678							IV		8	6
01234589							IV		8	7
01234789							IV		8	8
01236789							IV		8	9
02345679							IV		8	10
01234579							IV		8	11
01345679							IV		8	12
01234679							IV		8	13
01245679							IV		8	14
01234689							IV		8	15
01235789							IV		8	16
01345689							IV		8	17
01235689							IV		8	18
01245689							IV		8	19
01245789							IV		8	20
0123468T							IV		8	21
0123568T							IV		8	22
0123578T							IV		8	23
0124568T							IV		8	24
0124678T							IV		8	25
0124579T							IV		8	26
0124578T							IV		8	27
0134679T							IV		8	28
01235679							IV		8	29

Pitch Class Set	A (no tritone)			B (with tritone)						
	I (1/2)	III (1/2)	V	II a	II b 1/2	II b 3	IV (1/2)	VI		
012345678							IV		9	1
012345679							IV		9	2
012345689							IV		9	3
012345789							IV		9	4
012346789							IV		9	5
01234568T							IV		9	6
01234578T							IV		9	7
01234678T							IV		9	8
01235678T							IV		9	9
01234679T							IV		9	10
01235679T							IV		9	11
01245689T							IV		9	12

Chord successions

According to Hindemith, the complete material of conventional theory of harmony is contained in groups I, II and VI, except for a few exceptions of chords belonging to groups III and IV. "All possible chords may occur under conventional theory, but it accepts them only as structures resulting from strong melodic tendencies."[iii] Hindemith is expanding the theory but he accepts the results with objections. When chords from groups III and IV fight to be independent, he suggests to simply considering them as "non-existent".

[iii] Paul Hindemith, "The Craft of Musical Composition" (1937): 106.

Two types of movement are into play in a succession of chords. One has to do to the harmonic progression in which the relations of all the roots to their respective chords belong to the same sub-group (I_1 or I_2, or III_1, etc); the second is called *harmonic fluctuation* and it has to do with shift in harmonic gravity. Group I_1 is formed by triads, major and minor, in root position. Group 1_2 is formed by the same triads but in inversion. They are a little less stable than their counterparts in I_1. The tension increases as we proceed downward and from left to right in the table of chords. Thus, going from a chord belonging to group I to a chord of the group III, is an increase in tension. Increases and decreases in tension conform the harmonic fluctuation of a passage. In order to have harmonic fluctuation, chords of different groups are needed. Here are some examples:

The harmonic fluctuation is thus a sound aspect in composition. Even in passages that stay in the same root, there may be dramatic changes in harmonic value, providing interest. The progression is framed in a two-voices framework that provides the necessary contour for the chords.

In the next chapters, and until the final part on "Melody", Hindemith analyzes progressions with different chords, starting with progressions that have exclusively triads, that is, chords of group A.I. Then he proceeds with the other sonorities, with and without tritone, evaluating their characteristics. He suggests the use of a smooth degree progression if the chords have very varied harmonic weight, being part of very unequal groups. But, if chords belong to the same group, or

closely related groups, and the tension does not vary much, then the degree-progression may be more varied, with contrasts of several kind.

Finally, as practical application, Hindemith proposes an analysis at several levels, as shown in the following figure, which indicates the two-voice framework, the guide-tones, the root progression and the harmonic fluctuation.

Howard Hanson

Howard Hanson bases his study on the equally tempered chromatic scale, to make use of enharmonic equivalents and limit his work. A similar approach with *just* intonation would tend to infinite. Also, just intonation is possible for stringed instruments or voices but impossible for keyboard instruments or certain woodwinds and brass ones. Finally, equal

Howard Hanson

temperament offers the greatest simplicity of symbolism. The term *sonority*[iv], employed throughout the book, refers to the tones relationships, whether they involve harmony or melody.
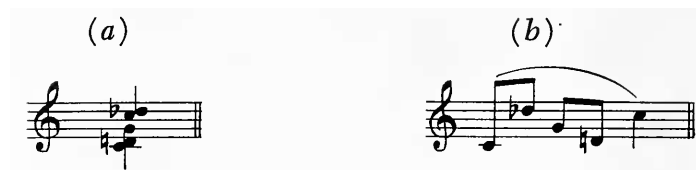
Many are the factors that make a sonority sound the way it does: relative degree of dissonance or consonance of its elements, disposition of the tones, duplications, timbre of the orchestration, etc. Moreover, some sonorities are associated with particular functions in tonality, or may have different importance according to the historic period and so on. Hanson focuses on the pitch content of the sonority, analyses its intervals and offers different combinations of simple intervals that can conform that particular group of tones. For example, C-E-G-B contains two P5, two M3, one m3 and one M7.

It may be explained as two P5 combined at the interval of a M3, or two M3 combined at the interval of a P5, or two triads[v] (C-E-G and E-G-B) combined, etc…

[iv] *Sonority*, *chord*, *set* are used as synonyms in this paper.
[v] The term *triad* is used to refer to any group of three tones, not just the major, minor, diminished and augmented triads.

This particular group of four tones has very well-known possible functions along music history. But for other sonorities, that may not always be the case. Hanson's work proposes then a thorough analysis of all the possible sonorities, how they are derived and which could be their connotations when used in compositions, either by masters or by composition students.

The analysis of intervals

For his analysis, Hanson does not consider doublings since the function of the sonority will not be altered by the number of times a note is duplicated; but he does consider *all* the intervals involved. For example, in the major triad C-E-G, he not considers the P5 between C and G, and the M3 between C and E, but also the m3 between E and G.

At the same time, he makes use of the circle of fifths to indicate that any note has interval relationships above and below it. For example, C has a fifth-relationship above it, C to G, and below it, C to F. Also, a major third-relationship above it, C to E, and below it, C to Ab, and so forth. The core of his theory is the simbology *pmnsdt*.

"p" represents the relationship of a perfect fifth above or below (what is actually a perfect fourth above) of the first tone.
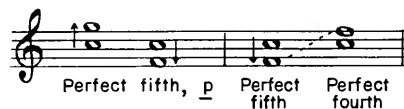
"m" a major third above or below (or a minor sixth above) of the first tone.
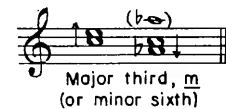
"n" a minor third above or below (or a major sixth above) of the first tone.
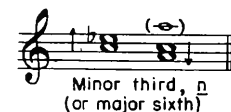
"s" a major second above or below (or a minor seventh above) of the first tone.
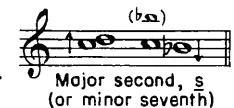
"d" a minor second above or below (or a major seventh above) of the first tone.
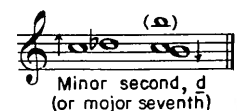
"t" a tritone above or below (augmented fourth or diminished fifth)
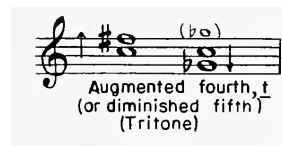
This table shows the intervals that each symbol represents:

symbol	interval (# semitones)	
p	7	5
m	4	8
n	3	9
s	2	10
d	1	11
t	6	

Every collection of sounds is given a label. Thus, the triad C-D-E containing two M2, and one M3, corresponds to ms^2, being "2" the number of that interval present in the sonority.

set	*p*		*m*		*n*		*s*		*d*		*t*	Hanson symbol
C-D-E	7	5	4	8	3	9	2	10	1	11	6	ms^2
			X				XX					

A more complex set, such as Eb-F-Ab-Bb-B, will have more intervals and correspondingly, a not so simple label. It possesses the six different type of interval, *pmnsdt*, some of them having more than one interval. Its label is $p^3mn^2s^2dt^{vi}$.

set	*p*		*m*		*n*		*s*		*d*		*t*	Hanson symbol
Eb-F-Ab-Bb-Cb	7	5	4	8	3	9	2	10	1	11	6	$p^3mn^2s^2dt$
	X	XX		X	XX		XX		X		X	

[vi] When there is only one interval of any type, it is represented by its letter only, for example *d* and not by d^1.

Once identified the symbols for all the trichords, Hanson starts referring to them as "*the triads pns…*", "*the triads ms²…*" etc.

The following is the complete list of Hanson symbols for the twelve possible triads, including also Forte's names[vii]:

Trichords	Forte	Hanson
012	3-1	sd^2
013	3-2	nsd
014	3-3	mnd
015	3-4	pmd
016	3-5	$pmdt$
024	3-6	ms^2
025	3-7	pns
026	3-8	mst
027	3-9	p^2s
036	3-10	n^2t
037	3-11	pmn
048	3-12	m^3

Involution

Hanson introduces the term "involution" to refer to the inverse ratio of an original sonority; that is, the projection downward from the lowest tone of the group maintaining the same intervals in the order they occur in the given chord. The involution is a sort of mirror of the original, and it always has the same intervals found in the original sonority.

[vii] Through this article there will be references to Allen Forte due to our familiarity with his theory, and certain advantages of his symbology.
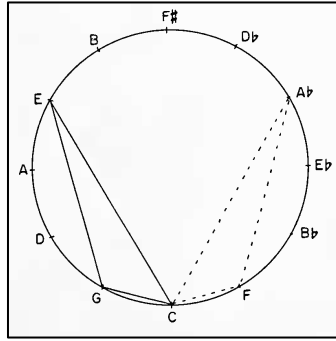
There are three types of involutions: a) simple, b) isometric and c) enharmonic.

a) <u>Simple</u>. The chord product of the involution has a different sound from the given chord. The major triad is a good example. Given the triad C-E-G, when the intervals are projected downward from C, the resulting chord, F-Ab-C, is a minor triad. C-E-G and its involution, F-Ab-C, both contain a P5, a M3 and a m3, and can be represented by *pmn*.

b) <u>Isometric</u>. In this involution, the given chord and its projection downward from the lowest tone have the same kind of sound. For example, C-E-G-B has the involution Db-F-Ab-C. Each of these tetrads is a major seventh chord, containing two P5, two M3, a m3 and a M7; and they are characterized by the symbols p^2m^2nd.
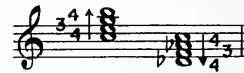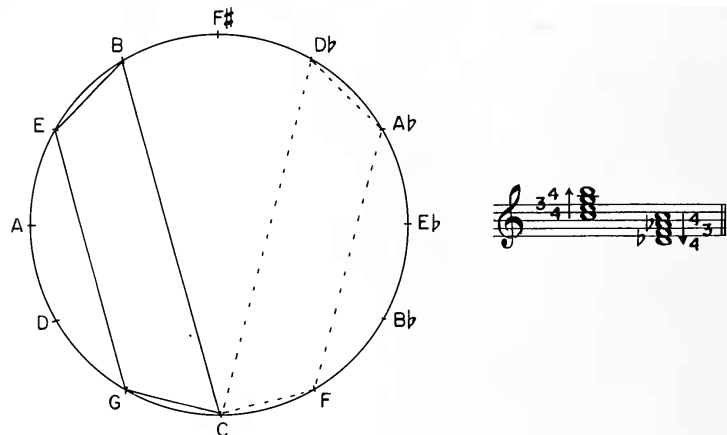
c) <u>Enharmonic</u>. In this type, the original sonority and its involution have the exact same tones in different octaves (except for the first tone). The augmented triad C-E-G#, for

example, when involuted, produces the augmented triad Fb-Ab-C. Due to equal temperament, Fb and Ab are equivalent to E and G#.
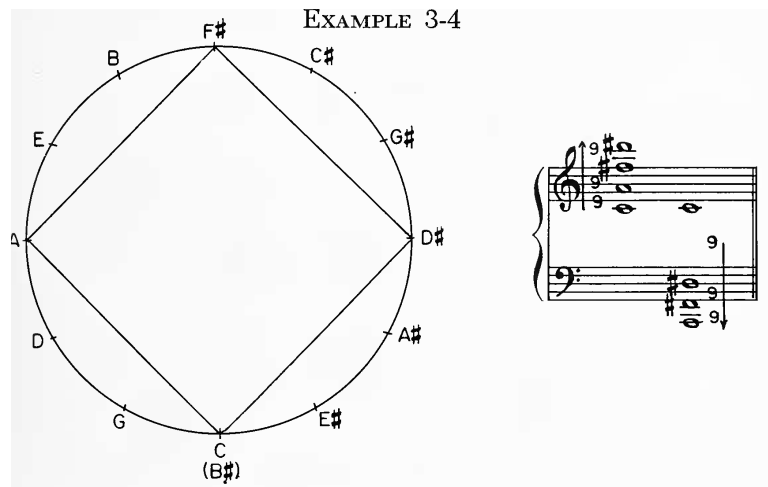
EXAMPLE 3-4

The combination of any sonority with its involution (for example C-E-G + F-Ab-C), produces a new chord (F-Ab-C-E-G in the example) that is isometric, reading the same intervals up and down, with the same amount of semitones in between.

Another example:

original: C – E – G – B
involution: Db – F – Ab – C
combination: Db – F – Ab – C – E – G – B
                       4   3   4   4   3   4

The axis of the involution does not necessarily have to be the lowest tone in the original sonority. For example, in the triad C-E-G, either E or G could be taken as axes. The combination of the original sonority with the involutions thus produced, generate different resulting chords that are all isometric in structure.

original: E – G – C
involution: E – G# – C#   (E as axis)
combination: C – C# – E – G – G#
                     1   3   3   1

original: G – C – E

involution: Bb – D – G   (G as axis)

combination: D – E – G – Bb – C

            2   3   3   2

It is interesting to mention that a chord can be isometric but not every disposition of its tones may show that. In other words, not every possible arrangement of the intervals is the same when read up or down. In the last example above, if we order the chord as:

E – G – Bb – C – D

3   3   2   2

The isometric character is not evident here. Out of the five possible dispositions of the tones, only the one that starts on D shows that the sonority is isometric because the order of semitones is the reverse of the order in the original chord.

The following are all isometric sonorities, though not necessarily presented in the disposition that shows this quality.

Involutions are not limited to those that contain a common tone with the given chord. Actually, in the case of the sonority C-E-G, any minor triad (and not only F-Ab-C) is a possible involution. Thus, the B minor triad, the D minor triad etc, are all involutions of C major triad.

Then the combination of the original and involution sonorities, as they do not share common tones, is a six-tone isometric sonority, instead of a five-tone one.
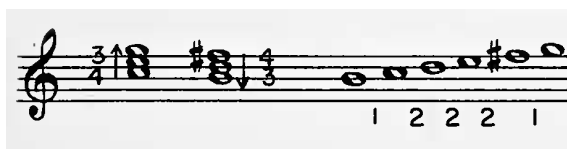
Isomeric sonorities

There are a few sonorities, which have the same components but are <u>not</u> involutions one of the other. They are called *isomeric*. It is the case for example of the tetrads C-E-F#-G and C-F#-G-Bb. Both contain one P5, one M3, one m3, one M2, one m2 and one tritone (*pmnsdt*) but they are not involution of each other; they both have their own involution.
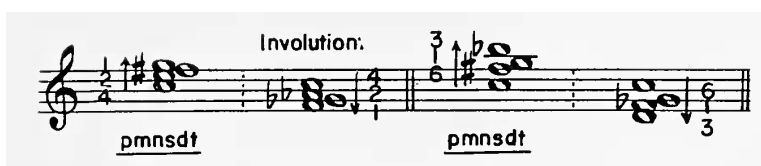
The six basic tonal series

It has been seen before that there are six types of internal relationship, considering such relationship up and down: the P5 and its inversion, the P4; the M3 and its inversion, the m9; the m3 and its inversion, the M6; the M2 and its inversion, the m7; the m2 and its inversion, the M7; and the tritone –augmented fourth and diminished fifth. Hence, we have the symbols *p*, *m*, *n*, *s*, *d* and *t* for each type. Given any collection of tones, it may have among its intervals, one interval that predominate, or a couple of interval that predominate, or the chord may be practically neutral in color. Thus, sonorities may fall into one of six great categories: perfect-fifth *types*, major-third *types*, minor-third *types*, etc. For example, among the six-tone sonorities (a total of forty nine), there are twenty-six dominated by one interval, twelve in which two intervals predominate, six with three intervals having equal strength and sis sonorities.

The simplest way of studying the relationship of tones is *to project* each of the basic six intervals (*p*, *m*, *n*, *s*, *d* and *t*). To *project* means to superimpose a series of similar intervals one above the other.

The only two intervals that are capable of yielding the twelve tones of the chromatic scale without repeating any tone (or its enharmonic) are the P5 and the m2.

Present and absent

The sound of a particular collection of tones –either presented as a melody or as harmony, and the way we perceive it, depends not only of the intervals present but also in the ones that are absent. For example, the pentad C-D-E-G-A (the familiar pentatonic scale) owes its particular sound to the preponderance of P5 along with the presence of M2, m3 and M3 (these intervals in decreasing progression), and also to the absence of the dissonant intervals of m2 (or M7) or tritone.
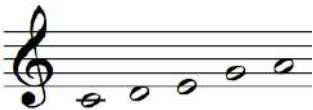
set	*p*		*m*		*n*		*s*		*d*		*t*	Hanson symbol
	7	5	4	8	3	9	2	10	1	11	6	
C-D-E-G-A	XX	XX	X		X	X	XXX					$p^4mn^2s^3$

Projection of the perfect fifth

Beginning with the tone C, we add first the perfect fifth G, and then the perfect fifth D, to produce the triad C-G-D or, reduced to the compass of an octave, C-D-G. This triad contains, in addition to the two fifths, the concomitant interval of the major second. It may be analyzed as p^2s.

It is called the *perfect-fifth* triad.

Afterwards, the addition of another P5, with the tone A, forms the *perfect-fifth* tetrad C-G-D-A, which contains three perfect fifths, two major seconds, and—for the first time in this series—a minor third, A to C.

Perfect Fifth Tetrad, p^3ns^2

Subsequent additions of another perfect fifth above will produce the *perfect-fifth* pentad (C-G-D-A-E), the *perfect-fifth* hexad (C-G-D-A-E-B), the *perfect-fifth* heptad (C-G-D-A-E-B-F#) and so on. All these groups can also be presented melodically, in a scalar formation that shows the interval –in number of semitones.

Perfect Fifth Heptad, $p^6m^3n^4s^5d^2t$

Perfect Fifth Pentad, $p^4mn^2s^3$

Perfect Fifth Hexad, $p^5m^2n^3s^4d$

Perfect Fifth Octad, $p^7m^4n^5s^6d^4t^2$
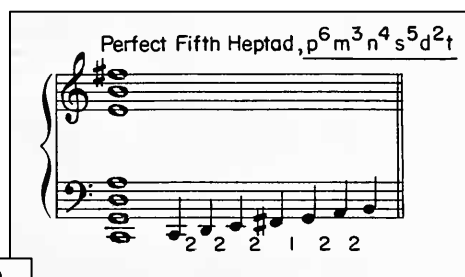
The heptad is the first sonority that contains all the different types of intervals, since it is the first time that the tritone appears. Hence, when the projection is carried over seven tones, no new intervals can be added. Moreover, on top of this loss of new materials, there is a decrease in the difference of the quantitative formation of the sonorities. The octad has the same amount of M3 and m2; the nonad has an equal number of M3, m3 and m2; the decad contains as many M3 as m3, M2 and m2. When the eleven- and twelve-tone sonorities are reached, there is no differentiation.

Perfect- Fifth Series			Hanson
	# tones	tones[viii]	
doad	2	07	p
triad	3	02**7**	$p^2\boldsymbol{s}$
tetrad	4	027**9**	$p^3\boldsymbol{n}s^2$
pentad	5	02**4**79	$p^4\boldsymbol{m}n^2s^3$
hexad	6	02479**e**	$p^5m^2n^3s^4\boldsymbol{d}$
heptad	7	024**6**79e	$p^6m^3n^4s^5d^2\boldsymbol{t}$
octad	8	0**1**24679e	$p^7m^4n^5s^6d^4t^2$
nonad	9	012467**8**9e	$p^8m^6n^6s^7d^6t^3$
decad	10	012**3**46789e	$p^9m^8n^8s^8d^8t^4$
undecad	11	012346789**te**	$p^{10}m^{10}n^{10}s^{10}d^{10}t^5$
duodecad	12	01234**5**6789te	$p^{12}m^{12}n^{12}s^{12}d^{12}t^6$

As a matter of fact, when sonorities are projected beyond the six-tone series they tend to lose their individuality. All seven-tone series, for example, contain all of the six basic intervals, and the difference in their proportion decreases as additional tones are added. Hanson sustains that the essential element of contrast in a composition is lost when the composer uses the twelve tones of the chromatic scale in a simple melodic or harmonic pattern. The music thus conceived tends to sound monochromatic with patterns that have little or no particular identity.

Harmonic-melodic material of the perfect-fifth hexad

For the reasons stated above, the six-tone combinations are the ones that offer the greatest number of different sonorities (or sets, or chords, or scale types –if presented in a scalar way). In order to have a study that is manageable, different dispositions of the same group of tones will be considered as one type of sonority. Thus, C-D-E-G-A is the same as E-G-A-C-D or D-E-G-A-C. And enharmonic tones are also considered different spelling of the same sound and hence, equivalent.

[viii] Numbers correspond with the tones in the order they are introduced. Dispositions are not necessarily their most compact versions, their prime forms.

The perfect-fifth hexad contains six types of triads, seven types of tetrads and three types of pentads. Some of the subsets are possible on notes other than C, and some have an involution.

	Type	Set	Forte	Involution	Isometric	Hanson
Perfect-Fifth Projection		C D G	3-9			p^2s
		C G A	3-7	C D A		pns
	Triads	C E G	3-11	A C E		pmn
		C G B	3-4	C E B		pmd
		C D E	3-6			ms^2
		B C D	3-2	A B C		nsd
		C D G A	4-23			p^3ns^2
		C D E G	4-22	A C D E		p^2mns^2
		C E G A	4-26		x^{ix}	p^2mn^2s
	Tetrads	C E G B	4-20		x	p^2m^2nd
		C D G B	4-14	C E A B		p^2mnsd
		G A B C	4-11	B C D E		$pmns^2d$
		A B C D	4-10		x	pn^2s^2d
		C D E G A	5-35			$p^4mn^2s^3$
	Pentads	C D E G B	5-27	C E G A B		$p^3m^2n^2s^2d$
		G A B C D	5-23	A B C D E		$p^3mn^2s^3d$

For each sonority, Hanson provides an explanation, in order to explain how it is produced as the combination of two transpositions of one same interval played together. For example, the tetrad A-B-C-D can be explained as two m3 at the distance of a M2, or two M2 at a m3.

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ix If one starts on the tone E and proceeds downward, EGAG, produces the identical tones.

Projection of the minor second

Likewise, beginning with the tone C, we can proceed by superimposing an interval of a m2, the tone C#, then another m2, D# etc. As said before, it is possible to obtain the twelve tones of the chromatic scale by this procedure, but the study limits to the hexad, C-C#-D-D#-E-F, which is known as the *minor-second* hexad.

Minor-Second Projection

Type	Set	Forte	Involution	Isometric	Hanson
Triads	CC#D	3-1			sd^2
	CC#D#	3-2	CDEb		nsd
	CC#E	3-3	CD#E		mnd
	CDbF	3-4	CEF		pmd
	CDE	3-6		x	ms^2
	CDF	3-7	CEbF		pns

Type	Set	Forte	Involution	Isometric	Hanson
Tetrads	CC#DD#	4-1		x	ns^2d^3
	CC#DE	4-2	CDD#E		mns^2d^2
	CC#D#E	4-3		x	mn^2sd^2
	CDbEF	4-7		x	pm^2nd^2
	CC#DF	4-4	CD#EF		$pmnsd^2$
	CDbEbF	4-11	CDEF		$pmns^2d$
	CDEbF	4-10		x	pn^2s^2d

Type	Set	Forte	Involution	Isometric	Hanson
Pentads	CC#DD#E	5-1		x	$pm^2n^2s^2d^3$
	CC#DEF	5-3	CC#D#EF		$pm^2n^2s^2d^3$
	CC#DEbF	5-2	CDD#EF		$pmn^2s^3d^3$

Projection of the major second

The projection of the interval of the major second is taken up to the six-tone sonority. One more major second would obtain a sound, B#, that is enharmonic of the first tone C.

The superposition of two M2 forms the major-second triad, ms^2; then the major-second tetrad, m^2s^3t; the major second pentad, $m^4s^4t^2$; and finally the major second hexad, the whole tone scale, $m^6s^6t^3$.

In order to produce a seven-tone scale, a *foreign* tone may be added, situated a perfect fifth from the first tone, C.
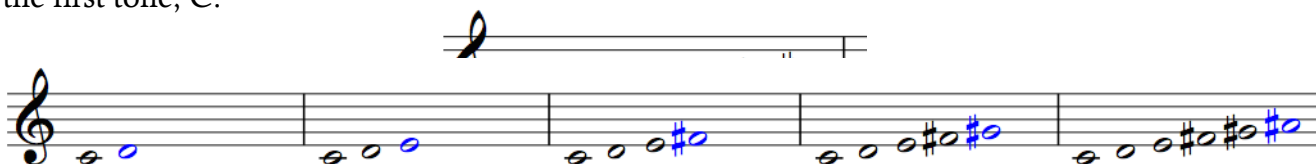
The inclusion of this note brings about the intervals missing so far, the m3, m2 and tritone.

											s
C	D										s
C	D	E									ms^2
C	D	E	F♯								m^2s^3t
C	D	E	F♯	G♯							$m^4s^4t^2$
C	D	E	F♯	G♯	A♯						$m^6s^6t^3$
C	D	E	F♯	G	G♯	A♯					$p^2m^6n^2s^6d^2t^3$
C	D	E	F♯	G	G♯	A	A♯				$p^4m^6n^4s^7d^4t^3$
C	D	E	F♯	G	G♯	A	A♯	B			$p^6m^7n^6s^8d^6t^3$
C	C♯	D	E	F♯	G	G♯	A	A♯	B		$p^8m^8n^8s^9d^8t^4$
C	C♯	D	D♯	E	F♯	G	G♯	A	A♯	B	$p^{10}m^{10}n^{10}s^{10}d^{10}t^5$
C	C♯	D	D♯	E	E♯	F♯	G	G♯	A	A♯ B	$p^{12}m^{12}n^{12}s^{12}d^{12}t^6$

The projection continues by superimposing a major second upon this new tone, obtaining A; then one more major second, the tone B, and so on, until the twelve tones of the chromatic scale appear. Like the perfect fifth and the second-major projections, it is useful to analyze all the sonorities that this new projection involves.

Besides the aforementioned major-second triad and major-second tetrad, there are other three- and four-tone sonorities: the triads C-E-G#, m^3, and C-D-F#, mst; and the tetrads C-D-E-G#, m^3s^2t, and C-E-F#-A#, $m^2s^2t^2$.

Projection of the minor third

The projection of the interval of the minor third second is limited to the four-tone sonority: C-Eb-Gb-Bbb. Another minor third would result in the tone Dbb, enharmonic of the first tone C.
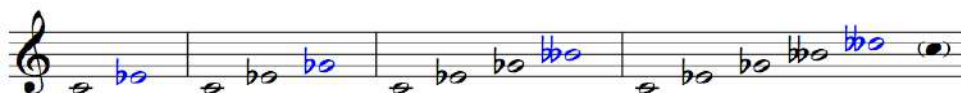
Again, in order to obtain sonorities beyond this limit, a foreign tone placed a perfect fifth above the first tone C, is added, the tone G. Then, the procedure continues with minor thirds, obtaining the tones Bb, Db and Fb.
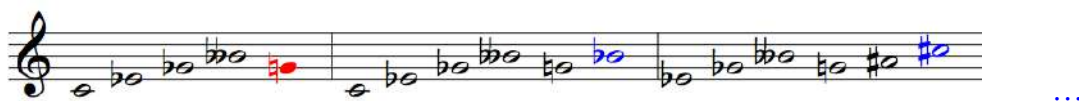
Now another tone situated a perfect fifth above G is added, D. Subsequent minor thirds upon this tone D, the tones F, Ab and C complete the chromatic aggregate.

The minor third projection produces:
- the triads
  - C-Eb-Gb and Eb-Gb-A (the diminished triad), n^2t
  - C-EbG and Eb-Gb-Bb (minor triad), pmn
  - C-G-A and Eb-Bb-C, pns
  - Gb-G-A and A-Bb-C, nsd
  - Eb-G-A and Gb-Bb-C, mst

- o Eb-Gb-G and Gb-A-Bb, *mnd*
- o C-Gb-G and Eb-A-Bb, *pdt*

- the tetrads
    - o C-Eb-Gb-A (the diminished seventh chord), n^4t^2
    - o C-Eb-G-Bb (the minor seventh chord, that is isometric), p^2mn^2s
    - o G-A-Bb-C (also isometric), pn^2s^2d
    - o C-Eb-Gb-Bb (the half-diminished seventh chord), pmn^2st
    - o C-Eb-Gb-G, pmn^2dt
    - o Gb-G-A-C, pn^2sdt
    - o Eb-Gb-G-A, mn^2sdt
    - o C- Gb-G-Bb, *pmnsdt*
    - o Eb-G-A-Bb, *pmnsdt*

- the pentads
    - o C-Eb-Gb-G-A, pmn^4sdt^2
    - o C-Eb-Gb-G-Bb, $p^2m^2n^3sdt$
    - o C-Eb-G-A-Bb, $p^2mn^3s^2dt$
    - o Eb-Gb-G-A-Bb, $pm^2n^3sd^2t$
    - o Gb-G-A-Bb-C, $pmn^3s^2d^2t$

As it can be seen, the complexities due to the inclusion of a foreign tone produce a greater number of different sonorities. From now on, the different sonorities that occur in the next projections will not be included in this text.

Projection of the major third

The major third projection suffers from the same limitation as major-second and minor-third projections. After superimposing two intervals of major third, obtaining E and G#, a new major third would produce B#, enharmonic of C. Therefore, it is necessary to include a foreign tone right here; one situated a perfect fifth above C, which is G. Two more major thirds can be placed above it, obtaining the tones B and D#, and once again, the possibilities run out. Another foreign

tone, D, is added and two more M3 can be placed above it to obtain the tones F# and A#. One last foreign tone, A, is necessary to continue further, obtaining the tones C# and E# that complete the aggregate.
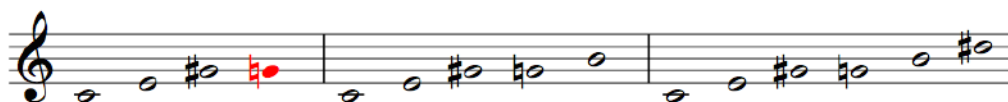
The study of the present sonorities –thirds, tetrads and pentads– will be limited to the hexad C-D#-E-G-G#-B, $p^3m^6n^3d^3$.

Projection of the tritone

The tritone is a strategic interval. Three of the six-tone series studied contain no tritone: the perfect-fifth, the minor-second and the major-third series. The other two do contain tritones, the major-second and minor-third series. The tritone itself is not useful as a unit to project because it runs out after superimposing two intervals of the same type: C-F#-B# (C).

So, the tritone projection must be formed by superimposing this interval upon the sclaes or sonorities that do not have themselves tritones; for example, the perfect-fifth series. The order of apparition of the tones will be then: C – F# – G – C# – D – G# – A – Eb – E – Bb – B – F. The hexad thus formed, has a preponderance of tritones, with perfect fifths and minor seconds as intervals of secondary importance. It is quite dissonant but possesses very interesting material.

Projections of triads

Once the possibilities of finding new sonorities with the projection of single intervals have been fully explored, it is time to project triad forms. There are five triads that contain three different intervals, and that exclude the tritone; they are the triads: *pmn*, *pns*, *pmd*, *mnd* and *nsd*. The procedure is to superimpose the same type of triad upon the perfect fifth of the first tone of the

given sonority. Thus, given the triad *pmn* C-E-G, another triad *pmn* is placed starting on G, obtaining the tones C-E-G + G-B-D, that is the pentad C-E-G-B-D, $p^3m^2n^2s^2d$, one of the fifths that appeared in the perfect-fifth projection.

Superimposing the same triad *pmn* upon the M3 of the first tone now, upon E, it is obtained: C-E-G + E-G#-B → C-E-G-G#-B, the pentad $p^3m^4n^3s^2d^2t$. Continuing similarly more five- and six-tone sonorities are obtained, which bring about triads, tetrads and pentads, some existing in the previous perfect-fifth and minor-second projections, and some new.

Other projections

After exhausting the possibilities of the projections with triads with no tritone, other combinations are explored. They include the projections of pair of intervals: m3+P5, m3+M3, m3+M2, m3+m2, P5+M3 and M3+m2. Again, to include here all the different sonorities and their connotations goes beyond the scope of this paper. However, let's observe how the different pentads appear along successive projections. Again, in order to easily verify the completeness of Hanson process, it is useful to refer the different pentads by their Forte names. For each pentad only it is indicated –with the blue boxes– the first time it appears. It is necessary to explore the projection of all those combinations to find the thirty-eight possibilities with five tones. The same occurs with smaller and larger groups.

Projections																	
	P5	m2	M2	m3	M3	TT	pmn -TT	pmn	pns	pmd	mn d	nsd	m3 +P5	m3 +M3	m3 +M2	m3+ m2	P5+ M3	M3 +m2
5-1		■																
5-2		■																
5-3		■																
5-4												■						
5-5																		■
5-6										■								
5-7						■												
5-8												■						
5-9																		
5-10				■														

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
5-11									X								
5-12													X				
5-13									X								
5-14																X	
5-15					X												
5-16				X													
5-17																	
5-18							X										
5-19						X											
5-20									X								
5-21					X												
5-22													X				
5-23	X																
5-24								X									
5-25				X													
5-26							X										
5-27	X																
5-28						X											
5-29								X									
5-30							X										
5-31				X													
5-32				X													
5-33			X														
5-34								X									
5-35	X																
5-36														X			
5-37																X	
5-38										X							

Complementary scales

Hanson devotes the last part of his book to talk about the theory of complementary scales. Every sonority has a *complementary* sonority –formed by the tones of the chromatic scale that are absent in the former– and that has the same type of intervallic analysis. A major triad, for example, will have a counterpart, a nine-tone chord, which is saturated of major thirds, and whose intervallic analysis shows a preponderance of perfect fifths, major thirds and minor thirds, the intervals present in its partner, the triad. This nonad is called the *projection* of the major triad, because it is like an expansion of the triad to the nine-tone order.

Chart

Harmonic Materials of Modern Music includes a chart called "The Projection and Interrelation of Sonorities in Equal Temperament". It is a complex web of relationships between sonorities, showing subsets and supersets, complementary sets, the Hanson symbols, and the order of semitones when the chord is presented as a scale. To comprehend this chart one has to proceed from the bottom top, and from right to left, until reaching the hexads. Then continue upward but from left to right until reaching the decads. Complementary sonorities are placed in opposite sides of the chart, according to a horizontal axis. Crosses indicate which subsets are included in the chords placed next to them. This is an example of how the chart looks:
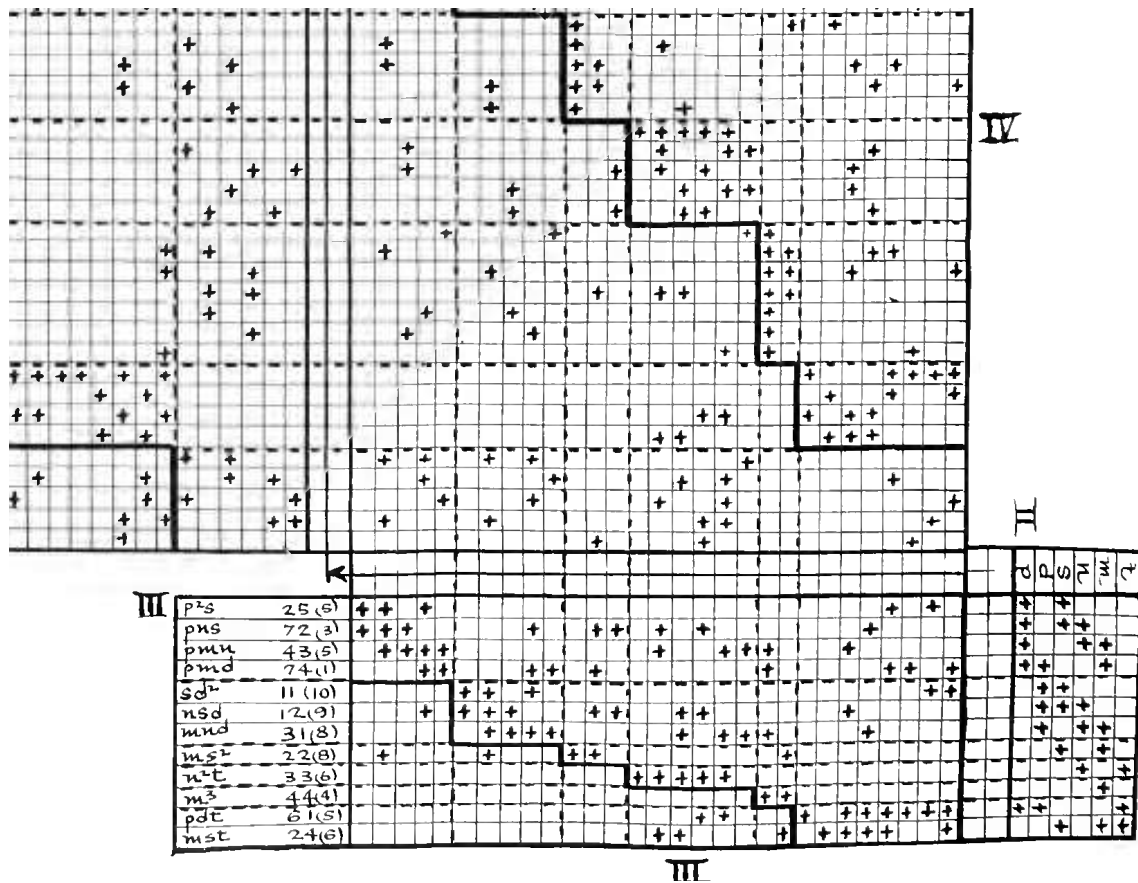
Comparison

Hindemith and Hanson's approach to the topic of harmonic materials is quite different. While one of the most important aspects Hindemith introduced was series 1 and 2 –with all the reasoning process that preceded them. The idea of having a hierarchy of intervals, both melodic and harmonic, is a useful tool for analysis and composition. And it could be the first step into something more rigorous that it is still to come. Hindemith attempts to be more inclusive in the materials feasible to be used, but he is still fundamentalist in keeping out the things he considers useless. He is far from considering *everything* as part of his harmonic language and he is honest about it.

The other important conclusion of Hindemith's work is his table of chords. It is a plausible attempt to organize the possible sonorities in a rational way. But he fell short. Even though Hindemith states that he had no intention to be exhaustive, the table is too far from being thorough and it is not completely rigorous.

Limitations of Hindemith's table of chords

Besides the design –that is a little confusing, especially for group B, the table of chords presents some inconsistencies.

a) some chords can belong to one subgroup or another depending on its voicing. For example, the trichord (027), when disposed as a quartal trichord, belongs to group V but when arranged in a close position, it belongs to group III.

b) The group VI says "Indeterminate. Tritone Predominating". The chords that appear there are basically two: the diminished triad and the fully diminished chord. The diminished triad has only one tritone while subgroup II.b.3 says "Containing more than one tritone". True, in this case, these chords have other intervals along with the tritone, but it is still confusing.

c) In the group II.b.3 Hindemith presents four chords and writes "etc.", suggesting that there are more chords like that. Two of the chords there presented are the same, Forte 4-25 and this tetrad is actually the only one (out of the three tetrads with more than one tritone as the

description of the subgroup says) that has no minor second[x]. And as for pentads, there is no pentad other than the one he presents as an example, (0248t), Forte 5-33, that fulfills the requirements of this subgroup. And the other chord presented as an example is the hexad (02468t), Forte 6-35, the whole tone scale. No other group of tones belongs to II.b.3. In Hanson's terms, we can conclude that the subtype II.b.3 is formed only by trichords, tetrachords, pentachords or hexachords derived from the projection of the major second.

d) To have subgroups 1. and 2. for groups III and IV, just based on where the root is positioned, in the bass or in a higher voice, is arguable, because it does not change the quality of those chords, given the complexity of them. The situation is identical with subgroups II.b1 and II.b2, which differentiate from each other in the position of the root (in the bass or not), but chords essentially remain the same, no matter the disposition.

e) Chords that belong to Group II contain tritones but lack minor seconds or major sevenths (intervals 1 and 11). A further subdivision considers the subtype II.a –with no major second either but with minor seventh. Hindemith shows here the seventh dominant chord, complete as a tetrad and incomplete –lacking the fifth. The tetrad also corresponds to the German Augmented sixth. Only one chord belongs to this subgroup: (047t), Forte 4-27, whose prime form is (0258). But if one does the involution of this chord, one obtains the half-diminished seventh chord. And this chord appears listed by Hindemith in the II.b.2.

Hanson

Hanson's work is systematic, methodical and comprehensive. The chart he arrives at, even though is complex and may not be grasped immediately by every student, it is consistent and it does not present flaws. On the other hand, he associates the sonorities with characteristic intervals, and presents them in a scalar way, what makes more sense than just the pitch class sets presented by Forte some fifteen years later. Besides, Hanson always relates de sonorities he is encountering through the projections with examples of real music. Thus, excerpts by Stravinski, Wagner, Copland, Schönberg, Vaughan-Williams, Bartok, Messiaen and many other contemporary composers, abound.

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[x] The other two are (0167), Forte 4-9 and (0369), Forte 4-28. The tetrad (0167) has two minor seconds and the tetrad (0369) is the fully diminished seventh chord, one of the two chords of group VI.

Hanson's writing style is direct and step-by-step, very pedagogical. Hindemith's prose, on the other hand, is very descriptive, a little literary sometimes; and it becomes somewhat subjective at moments.

The following are integrated tables with Hanson's symbols for all the triads, tetrads, pentads, hexads, heptads, octads and nonads, and their relationship with the different chords in Hindemith's classification. They also include the prime form for each group, Allen Forte's name and interval vector. Forte's theory provided a good departure point in order to cover all the possible sonorities. Hanson's symbols could then be easily calculated from the interval vector (and confirmed with his own labeling in his book) and finally each chord was classified according to Hindemith's criteria.

Triads

Pitch Class Set	Allen **Forte** — Forte's Name	Interval Vector	Howard **Hanson** — P5/P4, M3/m6, m3/M6, M2/m7, m2/M7, TT	Simplified notation	A (no tritone) I 1/2	A III 1/2	A V	B II a	B II b 1/2	B II b 3	B IV 1/2	B VI
012	3 1	2 1 0 0 0 0	$p^0 m^0 n^0 s^1 d^2 t^0$	$s\ d^2$								
013	3 2	1 1 1 0 0 0	$p^0 m^0 n^1 s^1 d^1 t^0$	$n\ s\ d$								
014	3 3	1 0 1 1 0 0	$p^0 m^1 n^1 s^0 d^1 t^0$	$m\ n\ d$								
015	3 4	1 0 0 1 1 0	$p^1 m^1 n^0 s^0 d^1 t^0$	$p\ m\ d$		III						
016	3 5	1 0 0 0 1 1	$p^1 m^0 n^0 s^0 d^1 t^1$	$p\ m\ d\ t$								
024	3 6	0 2 0 1 0 0	$p^0 m^1 n^0 s^2 d^0 t^0$	$m\ s^2$								
025	3 7	0 1 1 0 1 0	$p^1 m^0 n^1 s^1 d^0 t^0$	$p\ n\ s$		III						
026	3 8	0 1 0 1 0 1	$p^0 m^1 n^0 s^1 d^0 t^1$	$m\ s\ t$				II.a	II.b12			
027	3 9	0 1 0 0 2 0	$p^2 m^0 n^0 s^1 d^0 t^0$	$p^2\ s$								
036	3 10	0 0 2 0 0 1	$p^0 m^0 n^2 s^0 d^0 t^1$	$n^2\ t$								VI
037	3 11	0 0 1 1 1 0	$p^1 m^1 n^1 s^0 d^0 t^0$	$p\ m\ n$	I							
048	3 12	0 0 0 3 0 0	$p^0 m^3 n^0 s^0 d^0 t^0$	m^3			V					

Column group headers:
- Allen **Forte** — The Structure of Atonal Music (1977)
- Howard **Hanson** — Harmonic Materials of Modern Music (1960)
- Paul **Hindemith** — "The Craft of Musical Composition / Theory" (1942): A (no tritone), B (with tritone)

Interval Vector header: 1/11, 2/10, 3/6, 4/8, 5/7, 6
Hanson header: P5/P4, M3/m6, m3/M6, M2/m7, m2/M7, TT

Tetrads

Pitch Class Set	Forte's Name	Interval Vector						Howard Hanson (P5/P4, M3/m6, m3/M6, M2/m7, m2/M7, TT)	Simplified notation	A (no tritone)			B (with tritone)				
		1/11	2/10	3/6	4/8	5/7	6			I 1/2	III 1/2	V	II a	II b 1/2	II b 3	IV 1/2	VI
0123	4 1	3	2	1	0	0	0	$p^0 m^0 n^1 s^2 d^3 t^0$	$n\ s^2\ d^3$		III						
0124	4 2	2	2	1	1	0	0	$p^0 m^1 n^1 s^2 d^2 t^0$	$m\ n\ s^2\ d^2$		III						
0134	4 3	2	1	2	1	0	0	$p^0 m^1 n^2 s^1 d^2 t^0$	$m\ n^2\ s\ d^2$		III						
0125	4 4	2	1	1	1	1	0	$p^1 m^1 n^1 s^1 d^2 t^0$	$p\ m\ n\ s\ d^2$		III						
0126	4 5	2	1	0	1	1	1	$p^1 m^1 n^0 s^1 d^2 t^1$	$p\ m\ s\ d^2\ t$							IV	
0127	4 6	2	1	0	0	2	1	$p^2 m^0 n^0 s^1 d^2 t^1$	$p^2\ s\ d^2\ t$							IV	
0145	4 7	2	0	1	2	1	0	$p^1 m^2 n^1 s^0 d^2 t^0$	$p\ m^2\ n\ d^2$		III						
0156	4 8	2	0	0	1	2	1	$p^2 m^1 n^0 s^0 d^2 t^1$	$p^2\ m\ d^2\ t$							IV	
0167	4 9	2	0	0	0	2	2	$p^2 m^0 n^0 s^0 d^2 t^2$	$p^2\ d^2\ t^2$							IV	
0235	4 10	1	2	2	0	1	0	$p^1 m^0 n^2 s^2 d^1 t^0$	$p\ n^2\ s^2\ d$		III						
0135	4 11	1	0	1	1	1	0	$p^1 m^1 n^1 s^0 d^1 t^0$	$p\ m\ n\ d$		III						
0236	4 12	1	1	2	1	0	1	$p^0 m^1 n^2 s^1 d^1 t^1$	$m\ n^2\ s\ d\ t$							IV	
0136	4 13	1	1	2	0	1	1	$p^1 m^0 n^2 s^1 d^1 t^1$	$p\ n^2\ s\ d\ t$							IV	
0237	4 14	1	1	1	1	2	0	$p^2 m^1 n^1 s^1 d^1 t^0$	$p^2\ m\ n\ s\ d$		III						
0146	4 15	1	1	1	1	1	1	$p^1 m^1 n^1 s^1 d^1 t^1$	$p\ m\ n\ s\ d\ t$							IV	
0157	4 16	1	1	0	1	2	1	$p^2 m^1 n^0 s^1 d^1 t^1$	$p^2\ m\ s\ d\ t$							IV	
0347	4 17	1	0	2	2	1	0	$p^1 m^2 n^2 s^0 d^1 t^0$	$p\ m^2\ n^2\ d$		III						
0147	4 18	1	0	2	1	1	1	$p^1 m^1 n^2 s^0 d^1 t^1$	$p\ m\ n^2\ d\ t$							IV	
0148	4 19	1	0	1	3	1	0	$p^1 m^3 n^1 s^0 d^1 t^0$	$p\ m^3\ n\ d$		III						
0158	4 20	1	0	1	2	2	0	$p^2 m^2 n^1 s^0 d^1 t^0$	$p^2\ m^2\ n\ d$		III						
0246	4 21	0	3	0	2	0	1	$p^0 m^2 n^0 s^3 d^0 t^1$	$m^2\ s^3\ t$					II.b12			
0247	4 22	0	2	1	1	2	0	$p^2 m^1 n^1 s^2 d^0 t^0$	$p^2\ m\ n\ s^2$		III						
0257	4 23	0	2	1	0	3	0	$p^3 m^0 n^1 s^2 d^0 t^0$	$p^3\ n\ s^2$		III	V					
0248	4 24	0	2	0	3	0	1	$p^0 m^3 n^0 s^2 d^0 t^1$	$m^3\ s^2\ t$					II.b12			
0268	4 25	0	2	0	2	0	2	$p^0 m^2 n^0 s^2 d^0 t^2$	$m^2\ s^2\ t^2$						II.b3		
0358	4 26	0	1	2	1	2	0	$p^2 m^1 n^2 s^1 d^0 t^0$	$p^2\ m\ n^2\ s$		III						
0258	4 27	0	1	2	1	1	1	$p^1 m^1 n^2 s^1 d^0 t^1$	$p\ m\ n^2\ s\ t$				II.a				
0369	4 28	0	0	4	0	0	2	$p^0 m^0 n^4 s^0 d^0 t^2$	$n^4\ t^2$								VI
0137	4 29	1	1	1	1	1	1	$p^1 m^1 n^1 s^1 d^1 t^1$	$p\ m\ n\ s\ d\ t$							IV	

Pentads

Pitch Class Set	Forte's Name	\<-- Interval Vector --\> 1/11	2/10	3/6	4/8	5/7	6	Hanson (simplified)	Simplified notation	I 1/2	III 1/2	V	II a	II b1/2	II b3	IV 1/2	VI
01234	5 1	4	3	2	1	0	0	p m¹ n s³ d t	m n² s³ d⁴		III						
01235	5 2	3	3	2	1	1	0	p m¹ n s³ d t	p m n² s³ d³		III						
01245	5 3	3	2	2	2	1	0	p m² n s² d t	p m² n² s² d³		III						
01236	5 4	3	2	2	1	1	1	p m¹ n s² d t	p m n² s² d³ t							IV	
01237	5 5	3	2	1	1	2	1	p m¹ n s² d t	p² m n s² d³ t							IV	
01256	5 6	3	1	1	2	2	1	p m² n s¹ d t	p² m² n s d³ t							IV	
01267	5 7	3	1	0	1	3	2	p m¹ n s¹ d t	p³ m s d³ t							IV	
02346	5 8	2	3	2	2	0	1	p m² n s³ d t	m² n² s³ d² t							IV	
01246	5 9	2	3	1	2	1	1	p m² n s³ d t	p m² n s³ d² t							IV	
01346	5 10	2	2	3	1	1	1	p m¹ n s² d t	p m n³ s² d² t							IV	
02347	5 11	2	2	2	2	2	0	p m² n s² d t	p² m² n² s² d²		III						
01356	5 12	2	2	2	1	2	1	p m¹ n s² d t	p² m n² s² d² t							IV	
01248	5 13	2	2	1	3	1	1	p m³ n s² d t	p m³ n s² d² t							IV	
01257	5 14	2	2	1	1	3	1	p m¹ n s² d t	p³ m n s² d² t							IV	
01268	5 15	2	2	0	2	2	2	p m² n s² d t	p² m² s² d² t							IV	
01347	5 16	2	1	3	2	1	1	p m² n s¹ d t	p m² n³ s d² t							IV	
01348	5 17	2	1	2	3	2	0	p m³ n s¹ d t	p² m³ n² s d²		III						
01457	5 18	2	1	2	2	2	1	p m² n s¹ d t	p² m² n² s d² t							IV	
01367	5 19	2	1	2	1	2	2	p m¹ n s¹ d t	p² m n² s d² t							IV	
01378	5 20	2	1	1	2	3	1	p m² n s¹ d t	p³ m² n s d² t							IV	
01458	5 21	2	0	2	4	2	0	p m⁴ n s⁰ d t	p² m⁴ n² d²		III						
01478	5 22	2	0	2	3	2	1	p m³ n s⁰ d t	p² m³ n² d² t							IV	
02357	5 23	1	3	2	1	3	0	p m¹ n s³ d t	p³ m n² s³ d		III						
01357	5 24	1	3	1	2	2	1	p m² n s³ d t	p² m² n s³ d t							IV	
02358	5 25	1	2	3	1	2	1	p m¹ n s² d t	p² m n³ s² d t							IV	
02458	5 26	1	2	2	3	1	1	p m³ n s² d t	p m³ n² s² d t							IV	
01358	5 27	1	2	2	2	3	0	p m² n s² d t	p³ m² n² s² d		III						
02368	5 28	1	2	2	2	1	2	p m² n s² d t	p m² n² s² d t							IV	
01368	5 29	1	2	2	1	3	1	p m¹ n s² d t	p³ m n² s² d t							IV	
01468	5 30	1	2	1	3	2	1	p m³ n s² d t	p² m³ n s² d t							IV	
01369	5 31	1	1	4	1	1	2	p m¹ n s¹ d t	p m n⁴ s d t							IV	
01469	5 32	1	1	3	2	2	1	p m² n s¹ d t	p² m² n³ s d t							IV	
02468	5 33	0	4	0	4	0	2	p m⁴ n s⁴ d t	m⁴ s⁴ t						II.b3		
02469	5 34	0	3	2	2	2	1	p m² n s³ d t	p² m² n² s³ t					II.b12			
02479	5 35	0	3	2	1	4	0	p m¹ n s³ d t	p⁴ m n² s³		III						
01247	5 36	2	2	2	1	2	1	p m¹ n s² d t	p² m n² s² d² t							IV	
03458	5 37	2	1	2	3	2	0	p m³ n s¹ d t	p² m³ n² s d²		III						
01258	5 38	2	1	2	2	2	1	p m² n s¹ d t	p² m² n² s d² t							IV	

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Hexads

| Pitch Class Set | Forte's Name | Interval Vector | | | | | | Hanson P5/P4 | M3/m6 | m3/M6 | M2/m7 | m2/M7 | TT | Simplified notation | I 1/2 | III 1/2 | V | II a | II b1/2 | II b3 | IV 1/2 | VI |

Allen Forte — The Structure of Atonal Music (1977) · **Howard Hanson** — Harmonic Materials of Modern Music (1960) · **Paul Hindemith** — "The Craft of Musical Composition / Theory" (1942); A (no tritone), B (with tritone)

Pitch Class Set	Forte's Name	1/11	2/10	3/6	4/8	5/7	6	P5/P4	M3/m6	m3/M6	M2/m7	m2/M7	TT	Simplified notation	I 1/2	III 1/2	V	II a	II b1/2	II b3	IV 1/2	VI
012345	6 1	5	4	3	2	1	0	p^1	m^2	n^3	s^4	d^5	t^0	p m n s d		III						
012346	6 2	4	4	3	2	1	1	p^1	m^2	n^3	s^4	d^4	t^1	p m n s d t							IV	
012356	6 3	4	3	3	2	2	1	p^2	m^2	n^3	s^3	d^4	t^1	p m n s d t							IV	
012456	6 4	4	3	2	3	2	1	p^2	m^3	n^2	s^3	d^4	t^1	p m n s d t							IV	
012367	6 5	4	2	2	2	3	2	p^3	m^2	n^2	s^2	d^4	t^2	p m n s d t							IV	
012567	6 6	4	2	1	2	4	2	p^4	m^2	n^1	s^2	d^4	t^2	p m n s d t							IV	
012678	6 7	4	2	0	2	4	3	p^4	m^2	n^0	s^2	d^4	t^3	p m s d t							IV	
023457	6 8	3	4	3	2	3	0	p^3	m^2	n^3	s^4	d^3	t^0	p m n s d		III						
012357	6 9	3	4	2	2	3	1	p^3	m^2	n^2	s^4	d^3	t^1	p m n s d t							IV	
013457	6 10	3	3	3	3	2	1	p^2	m^3	n^3	s^3	d^3	t^1	p m n s d t							IV	
012457	6 11	3	3	3	2	3	1	p^3	m^2	n^3	s^3	d^3	t^1	p m n s d t							IV	
012467	6 12	3	3	2	2	3	2	p^3	m^2	n^2	s^3	d^3	t^2	p m n s d t							IV	
013467	6 13	3	2	4	2	2	2	p^2	m^2	n^4	s^2	d^3	t^2	p m n s d t							IV	
013458	6 14	3	2	3	4	3	0	p^3	m^4	n^3	s^2	d^3	t^0	p m n s d		III						
012458	6 15	3	2	3	4	2	1	p^2	m^4	n^3	s^2	d^3	t^1	p m n s d t							IV	
014568	6 16	3	2	2	4	3	1	p^3	m^4	n^2	s^2	d^3	t^1	p m n s d t							IV	
012478	6 17	3	2	2	3	3	2	p^3	m^3	n^2	s^2	d^3	t^2	p m n s d t							IV	
012578	6 18	3	2	2	2	4	2	p^4	m^2	n^2	s^2	d^3	t^2	p m n s d t							IV	
013478	6 19	3	1	3	4	3	1	p^3	m^4	n^3	s^1	d^3	t^1	p m n s d t							IV	
014589	6 20	3	0	3	6	3	0	p^3	m^6	n^3	s^0	d^3	t^0	p m n d		III						
023468	6 21	2	4	2	4	1	2	p^1	m^4	n^2	s^4	d^2	t^2	p m n s d t							IV	
012468	6 22	2	4	1	4	2	2	p^2	m^4	n^1	s^4	d^2	t^2	p m n s d t							IV	
023568	6 23	2	3	4	2	2	2	p^2	m^2	n^4	s^3	d^2	t^2	p m n s d t							IV	
013468	6 24	2	3	3	3	3	1	p^3	m^3	n^3	s^3	d^2	t^1	p m n s d t							IV	
013568	6 25	2	3	3	2	4	1	p^4	m^2	n^3	s^3	d^2	t^1	p m n s d t							IV	
013578	6 26	2	3	2	3	4	1	p^4	m^3	n^2	s^3	d^2	t^1	p m n s d t							IV	
013469	6 27	2	2	5	2	2	2	p^2	m^2	n^5	s^2	d^2	t^2	p m n s d t							IV	
013569	6 28	2	2	4	3	2	2	p^2	m^3	n^4	s^2	d^2	t^2	p m n s d t							IV	
013689	6 29	2	2	4	2	3	2	p^3	m^2	n^4	s^2	d^2	t^2	p m n s d t							IV	
013679	6 30	2	2	4	2	2	3	p^2	m^2	n^4	s^2	d^2	t^3	p m n s d t							IV	
013589	6 31	2	2	3	4	3	1	p^3	m^4	n^3	s^2	d^2	t^1	p m n s d t							IV	
024579	6 32	1	4	3	2	5	0	p^5	m^2	n^3	s^4	d^1	t^0	p m n s d		III						
023579	6 33	1	4	3	2	4	1	p^4	m^2	n^3	s^4	d^1	t^1	p m n s d t							IV	
013579	6 34	1	4	2	4	2	2	p^2	m^4	n^2	s^4	d^1	t^2	p m n s d t							IV	

Pitch Class Set	Forte's Name	1/11	2/10	3/6	4/8	5/7	6	Hanson notation	Simplified notation	I 1/2	III 1/2	V	II a	II b1/2	II b3	IV 1/2	VI
02468T	6 35	0	6	0	6	0	3	p⁰m⁶n⁰s⁶d⁰t³	m s t						II.b3		
012347	6 36	4	3	3	2	2	1	p²m²n³s³d⁴t¹	p m n s d t							IV	
012348	6 37	4	3	2	3	2	1	p²m³n²s³d⁴t¹	p m n s d t							IV	
012378	6 38	4	2	1	2	4	2	p⁴m²n¹s²d⁴t²	p m n s d t							IV	
023458	6 39	3	3	3	3	2	1	p²m³n³s³d³t¹	p m n s d t							IV	
012358	6 40	3	3	3	2	3	1	p³m²n³s³d³t¹	p m n s d t							IV	
012368	6 41	3	3	2	2	3	2	p³m²n²s³d³t²	p m n s d t							IV	
012369	6 42	3	2	4	2	2	2	p²m²n⁴s²d³t²	p m n s d t							IV	
012568	6 43	3	2	2	3	3	2	p³m³n²s²d³t²	p m n s d t							IV	
012569	6 44	3	1	3	4	3	1	p³m⁴n³s¹d³t¹	p m n s d t							IV	
023469	6 45	2	3	4	2	2	2	p²m²n⁴s³d²t²	p m n s d t							IV	
012469	6 46	2	3	3	3	3	1	p³m³n³s³d²t¹	p m n s d t							IV	
012479	6 47	2	3	3	2	4	1	p⁴m²n³s³d²t¹	p m n s d t							IV	
012579	6 48	2	3	2	3	4	1	p⁴m³n²s³d²t¹	p m n s d t							IV	
013479	6 49	2	2	4	3	2	2	p²m³n⁴s²d²t²	p m n s d t							IV	
014679	6 50	2	2	4	2	3	2	p³m²n⁴s²d²t²	p m n s d t							IV	

Heptads

Pitch Class Set	Allen **Forte** *The Structure of Atonal Music* (1977) Forte's Name	Interval Vector 1/11	2/10	3/6	4/8	5/7	6	Howard **Hanson** *Harmonic Materials of Modern Music* (1960) P5/P4, M3/m6, m3/M6, M2/m7, m2/M7, TT	Simplified notation	Paul **Hindemith** *"The Craft of Musical Composition / Theory"* (1942) A (no tritone) I 1/2	III 1/2	V	B (with tritone) II a	b1/2	b3	IV 1/2	VI
0123456	7 1	6	5	4	3	2	1	p²m³n⁴s⁵d⁶t¹	p m n s d t							IV	
0123457	7 2	5	5	4	3	3	1	p³m³n⁴s⁵d⁵t¹	p m n s d t							IV	
0123458	7 3	5	4	4	4	3	1	p³m⁴n⁴s⁴d⁵t¹	p m n s d t							IV	
0123467	7 4	5	4	4	3	3	2	p³m³n⁴s⁴d⁵t²	p m n s d t							IV	
0123567	7 5	5	4	3	3	4	2	p⁴m³n³s⁴d⁵t²	p m n s d t							IV	
0123478	7 6	5	3	3	4	4	2	p⁴m⁴n³s³d⁵t²	p m n s d t							IV	
0123678	7 7	5	3	2	3	5	3	p⁵m³n²s³d⁵t³	p m n s d t							IV	
0234568	7 8	4	5	4	4	2	2	p²m⁴n⁴s⁵d⁴t²	p m n s d t							IV	
0123468	7 9	4	5	3	4	3	2	p³m⁴n³s⁵d⁴t²	p m n s d t							IV	
0123469	7 10	4	4	5	3	3	2	p³m³n⁵s⁴d⁴t²	p m n s d t							IV	
0134568	7 11	4	4	4	4	4	1	p⁴m⁴n⁴s⁴d⁴t¹	p m n s d t							IV	
0123479	7 12	4	4	4	3	4	2	p⁴m³n⁴s⁴d⁴t²	p m n s d t							IV	
0124568	7 13	4	4	3	5	3	2	p³m⁵n³s⁴d⁴t²	p m n s d t							IV	
0123578	7 14	4	4	3	3	5	2	p⁵m³n³s⁴d⁴t²	p m n s d t							IV	
0124678	7 15	4	4	2	4	4	3	p⁴m⁴n²s⁴d⁴t³	p m n s d t							IV	
0123569	7 16	4	3	5	4	3	2	p³m⁴n⁵s³d⁴t²	p m n s d t							IV	
0124569	7 17	4	3	4	5	4	1	p⁴m⁵n⁴s³d⁴t¹	p m n s d t							IV	

Pitch Class Set	Forte	Interval Vector	Hanson (P5 M3 m3 M2 m2 TT)	Simplified notation	I	III	V	II a	II b	II b	IV	VI
0123589	7 18	4 3 4 4 4 2	$p^4 m^4 n^4 s^3 d^4 t^2$	p m n s d t							**IV**	
0123679	7 19	4 3 4 3 4 3	$p^4 m^3 n^4 s^3 d^4 t^3$	p m n s d t							**IV**	
0124789	7 20	4 3 3 4 5 2	$p^5 m^4 n^3 s^3 d^4 t^2$	p m n s d t							**IV**	
0124589	7 21	4 2 4 6 4 1	$p^4 m^6 n^4 s^2 d^4 t^1$	p m n s d t							**IV**	
0125689	7 22	4 2 4 5 4 2	$p^4 m^5 n^4 s^2 d^4 t^2$	p m n s d t							**IV**	
0234579	7 23	3 5 4 3 5 1	$p^5 m^3 n^4 s^5 d^3 t^1$	p m n s d t							**IV**	
0123579	7 24	3 5 3 4 4 2	$p^4 m^4 n^3 s^5 d^3 t^2$	p m n s d t							**IV**	
0234679	7 25	3 4 5 3 4 2	$p^4 m^3 n^5 s^4 d^3 t^2$	p m n s d t							**IV**	
0134579	7 26	3 4 4 5 3 2	$p^3 m^5 n^4 s^4 d^3 t^2$	p m n s d t							**IV**	
0124579	7 27	3 4 4 4 5 1	$p^5 m^4 n^4 s^4 d^3 t^1$	p m n s d t							**IV**	
0135679	7 28	3 4 4 4 3 3	$p^3 m^4 n^4 s^4 d^3 t^3$	p m n s d t							**IV**	
0124679	7 29	3 4 4 3 5 2	$p^5 m^3 n^4 s^4 d^3 t^2$	p m n s d t							**IV**	
0124689	7 30	3 4 3 5 4 2	$p^4 m^5 n^3 s^4 d^3 t^2$	p m n s d t							**IV**	
0134679	7 31	3 3 6 3 3 3	$p^3 m^3 n^6 s^3 d^3 t^3$	p m n s d t							**IV**	
0134689	7 32	3 3 5 4 4 2	$p^4 m^4 n^5 s^3 d^3 t^2$	p m n s d t							**IV**	
012468T	7 33	2 6 2 6 2 3	$p^2 m^6 n^2 s^6 d^2 t^3$	p m n s d t							**IV**	
013468T	7 34	2 5 4 4 4 2	$p^4 m^4 n^4 s^5 d^2 t^2$	p m n s d t							**IV**	
013568T	7 35	2 5 4 3 6 1	$p^6 m^3 n^4 s^5 d^2 t^1$	p m n s d t							**IV**	
0123568	7 36	4 4 4 3 4 2	$p^4 m^3 n^4 s^4 d^4 t^2$	p m n s d t							**IV**	
0134578	7 37	4 3 4 5 4 1	$p^4 m^5 n^4 s^3 d^4 t^1$	p m n s d t							**IV**	
0124578	7 38	4 3 4 4 4 2	$p^4 m^4 n^4 s^3 d^4 t^2$	p m n s d t							**IV**	

Octads

Pitch Class Set	Allen **Forte** The Structure of Atonal Music (1977) Forte's Name	Interval Vector 1/11 2/10 3/6 4/8 5/7 6	Howard **Hanson** Harmonic Materials of Modern Music (1960) P5/P4 M3/m6 m3/M6 M2/m7 m2/M7 TT	Simplified notation	Paul **Hindemith** "The Craft of Musical Composition / Theory" (1942) A (no tritone) I 1/2	III 1/2	V	B (with tritone) II a	b 1/2	b 3	IV 1/2	VI
01234567	8 1	7 6 5 4 4 2	$p^4 m^4 n^5 s^6 d^7 t^2$	$p^4 m^4 n^5 s^6 d^7 t^2$							**IV**	
01234568	8 2	6 6 5 5 4 2	$p^4 m^5 n^5 s^6 d^6 t^2$	$p^4 m^5 n^5 s^6 d^6 t^2$							**IV**	
01234569	8 3	6 5 6 5 4 2	$p^4 m^5 n^6 s^5 d^6 t^2$	$p^4 m^5 n^6 s^5 d^6 t^2$							**IV**	
01234578	8 4	6 5 5 5 5 2	$p^5 m^5 n^5 s^5 d^6 t^2$	$p^5 m^5 n^5 s^5 d^6 t^2$							**IV**	
01234678	8 5	6 5 4 5 5 3	$p^5 m^5 n^4 s^5 d^6 t^3$	$p^5 m^5 n^4 s^5 d^6 t^3$							**IV**	
01235678	8 6	6 5 4 4 6 3	$p^6 m^4 n^4 s^5 d^6 t^3$	$p^6 m^4 n^4 s^5 d^6 t^3$							**IV**	
01234589	8 7	6 4 5 6 5 2	$p^5 m^6 n^5 s^4 d^6 t^2$	$p^5 m^6 n^5 s^4 d^6 t^2$							**IV**	
01234789	8 8	6 4 4 5 6 3	$p^6 m^5 n^4 s^4 d^6 t^3$	$p^6 m^5 n^4 s^4 d^6 t^3$							**IV**	
01236789	8 9	6 4 4 4 6 4	$p^6 m^4 n^4 s^4 d^6 t^4$	$p^6 m^4 n^4 s^4 d^6 t^4$							**IV**	
02345679	8 10	5 6 6 4 5 2	$p^5 m^4 n^6 s^6 d^5 t^2$	$p^5 m^4 n^6 s^6 d^5 t^2$							**IV**	
01234579	8 11	5 6 5 5 5 2	$p^5 m^5 n^5 s^6 d^5 t^2$	$p^5 m^5 n^5 s^6 d^5 t^2$							**IV**	
01345679	8 12	5 5 6 5 4 3	$p^4 m^5 n^6 s^5 d^5 t^3$	$p^4 m^5 n^6 s^5 d^5 t^3$							**IV**	
01234679	8 13	5 5 6 4 5 3	$p^5 m^4 n^6 s^5 d^5 t^3$	$p^5 m^4 n^6 s^5 d^5 t^3$							**IV**	
01245679	8 14	5 5 5 5 6 2	$p^6 m^5 n^5 s^5 d^5 t^2$	$p^6 m^5 n^5 s^5 d^5 t^2$							**IV**	

Pitch Class Set	Forte's Name		Interval Vector						Hanson notation	Simplified notation	I 1/2	III 1/2	V	a	b 1/2	b 3	IV 1/2	VI
01234689	8	15	5	5	5	5	5	3	$p^5 m^5 n^5 s^5 d^5 t^3$	$p^5 m^5 n^5 s^5 d^5 t^3$							**IV**	
01235789	8	16	5	5	4	5	6	3	$p^6 m^5 n^4 s^5 d^5 t^3$	$p^6 m^5 n^4 s^5 d^5 t^3$							**IV**	
01345689	8	17	5	4	6	6	5	2	$p^5 m^6 n^6 s^4 d^5 t^2$	$p^5 m^6 n^6 s^4 d^5 t^2$							**IV**	
01235689	8	18	5	4	6	5	5	3	$p^5 m^5 n^6 s^4 d^5 t^3$	$p^5 m^5 n^6 s^4 d^5 t^3$							**IV**	
01245689	8	19	5	4	5	7	5	2	$p^5 m^7 n^5 s^4 d^5 t^2$	$p^5 m^7 n^5 s^4 d^5 t^2$							**IV**	
01245789	8	20	5	4	5	6	6	2	$p^6 m^6 n^5 s^4 d^5 t^2$	$p^6 m^6 n^5 s^4 d^5 t^2$							**IV**	
0123468T	8	21	4	7	4	6	4	3	$p^4 m^6 n^4 s^7 d^4 t^3$	$p^4 m^6 n^4 s^7 d^4 t^3$							**IV**	
0123568T	8	22	4	6	5	5	6	2	$p^6 m^5 n^5 s^5 d^4 t^2$	$p^6 m^5 n^5 s^6 d^4 t^2$							**IV**	
0123578T	8	23	4	6	5	4	7	2	$p^7 m^4 n^5 s^6 d^4 t^2$	$p^7 m^4 n^5 s^6 d^4 t^2$							**IV**	
0124568T	8	24	4	6	4	7	4	3	$p^4 m^7 n^4 s^6 d^4 t^3$	$p^4 m^7 n^4 s^6 d^4 t^3$							**IV**	
0124678T	8	25	4	6	4	6	4	4	$p^4 m^6 n^4 s^6 d^4 t^4$	$p^4 m^6 n^4 s^6 d^4 t^4$							**IV**	
0124579T	8	26	4	5	6	5	6	2	$p^6 m^5 n^6 s^5 d^4 t^2$	$p^6 m^5 n^6 s^5 d^4 t^2$							**IV**	
0124578T	8	27	4	5	6	5	5	3	$p^5 m^5 n^6 s^5 d^4 t^3$	$p^5 m^5 n^6 s^5 d^4 t^3$							**IV**	
0134679T	8	28	4	4	8	4	4	4	$p^4 m^4 n^8 s^4 d^4 t^4$	$p^4 m^4 n^8 s^4 d^4 t^4$							**IV**	
01235679	8	29	5	5	5	5	5	3	$p^5 m^5 n^5 s^5 d^5 t^3$	$p^5 m^5 n^5 s^5 d^5 t^3$							**IV**	
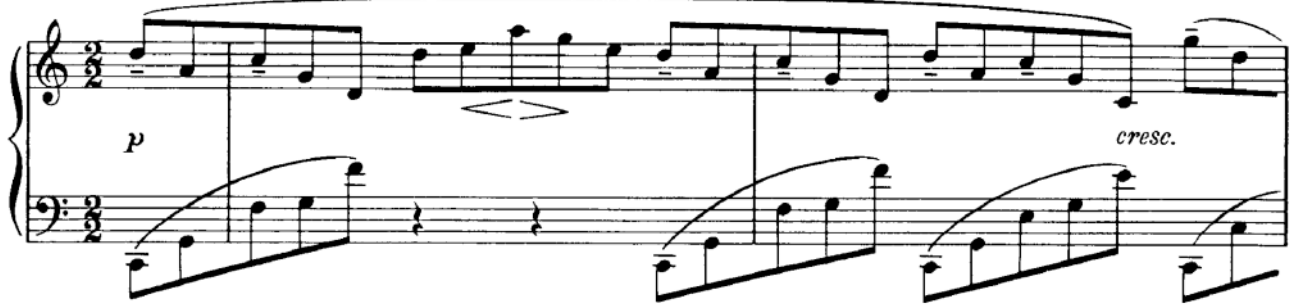
Nonads

Pitch Class Set	Allen **Forte** — The Structure of Atonal Music (1977)									Howard **Hanson** — Harmonic Materials of Modern Music (1960)							Paul **Hindemith** — "The Craft of Musical Composition / Theory" (1942)								
	Forte's Name		Interval Vector							P5/P4, M3/m6, m3/M6, M2/m7, m2/M7, TT						Simplified notation	A (no tritone)			B (with tritone)				VI
			1 / 11	2 / 10	3 / 6	4 / 8	5 / 7	6									I 1/2	III 1/2	V	a	b 1/2	b 3	IV 1/2	
012345678	9	1	8	7	6	6	6	3		$p^6 m^6 n^6 s^7 d^8 t^3$							$p^6 m^6 n^6 s^7 d^8 t^3$							**IV**	
012345679	9	2	7	7	7	6	6	3		$p^6 m^6 n^7 s^7 d^7 t^3$							$p^6 m^6 n^7 s^7 d^7 t^3$							**IV**	
012345689	9	3	7	6	7	7	6	3		$p^6 m^7 n^7 s^6 d^7 t^3$							$p^6 m^7 n^7 s^6 d^7 t^3$							**IV**	
012345789	9	4	7	6	6	7	7	3		$p^7 m^7 n^6 s^6 d^7 t^3$							$p^7 m^7 n^6 s^6 d^7 t^3$							**IV**	
012346789	9	5	7	6	6	6	7	4		$p^7 m^6 n^6 s^6 d^7 t^4$							$p^7 m^6 n^6 s^6 d^7 t^4$							**IV**	
01234568T	9	6	6	8	6	7	6	3		$p^6 m^7 n^6 s^8 d^6 t^3$							$p^6 m^7 n^6 s^8 d^6 t^3$							**IV**	
01234578T	9	7	6	7	7	6	7	3		$p^7 m^6 n^7 s^7 d^6 t^3$							$p^7 m^6 n^7 s^7 d^6 t^3$							**IV**	
01234678T	9	8	6	7	6	7	6	4		$p^6 m^7 n^6 s^7 d^6 t^4$							$p^6 m^7 n^6 s^7 d^6 t^4$							**IV**	
01235678T	9	9	6	7	6	6	8	3		$p^8 m^6 n^6 s^7 d^6 t^3$							$p^8 m^6 n^6 s^7 d^6 t^3$							**IV**	
01234679T	9	10	6	6	8	6	6	4		$p^6 m^6 n^8 s^6 d^6 t^4$							$p^6 m^6 n^8 s^6 d^6 t^4$							**IV**	
01235679T	9	11	6	6	7	7	7	3		$p^7 m^7 n^7 s^6 d^6 t^3$							$p^7 m^7 n^7 s^6 d^6 t^3$							**IV**	
01245689T	9	12	6	6	6	9	6	3		$p^6 m^9 n^6 s^6 d^6 t^3$							$p^6 m^9 n^6 s^6 d^6 t^3$							**IV**	

Analysis

In order to show both Hanson and Hindemith's theories in action, I analyzed the first prelude of "Twenty-Four Préludes", opus 11, by Alexander Scriabin, composed in 1895.
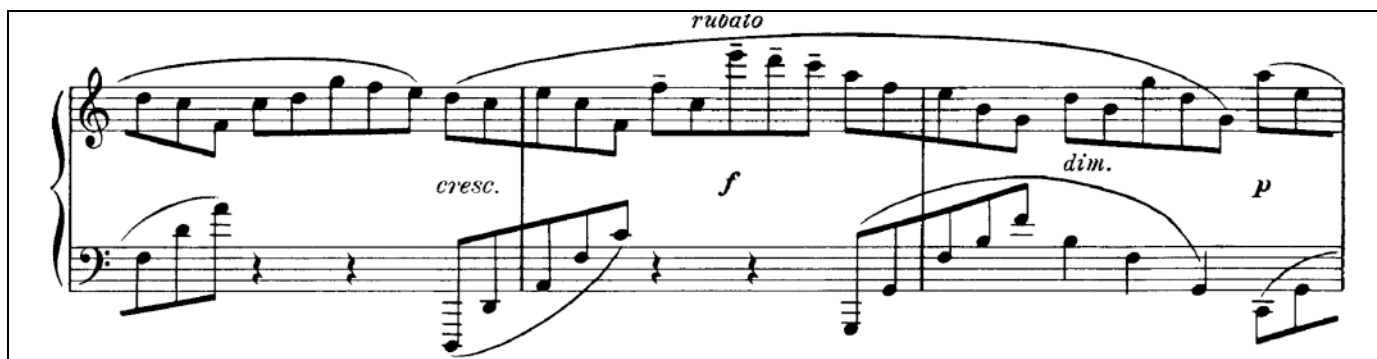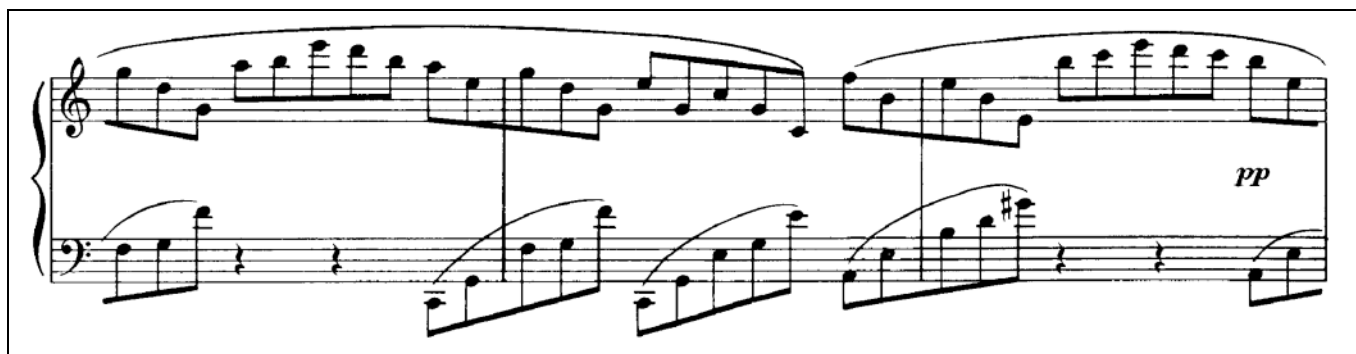

Sonority	C - D - E - F - G - A		
Root	(C) G		C	C
Forte Name	6-32		
Hindemith Group	III		III
Hanson symbol	$p^5m^2n^3s^4d$		

Sonority	F - G - A - B♭ - C - D		C - D - E - F - G - A	
Root	...(C)	F	D	A
Forte Name	6-32		6-32	
Hindemith Group	III		III	
Hanson symbol	$p^5m^2n^3s^4d$			

Sonority	C - D - E - F - G - A		C - D - E - F - G - A - B		
Root		D	G		(C) G
Forte Name	6-32		7-35		
Hindemith Group	III		IV		
Hanson symbol	p⁵m²n³s⁴d		p⁶m³n⁴s⁵d²t		

Sonority	C - D - E - F - G - A		A - B - C - D - E - F - G♯	
Root	(C) G	C	(A) E	
Forte Name	6-32		7-32	
Hindemith Group	III		IV	
Hanson symbol	p⁵m²n³s⁴d		p⁴m⁴n⁵s³d³t²	

Sonority			D♯ - E - F♯ - G - A - B - C		C♯ - D
Root	A		D♯	E	C♯
Forte Name	7-32		7-32		7-32
Hindemith Group	IV		IV		IV
Hanson symbol			p⁵m²n³s⁴d		

Sonority	- E - F - G - A - B♭	C - D - E - F - A♭	C - D - E - G	D - E - F♯ - A♭ - C	C - D - E - F - A	
Root	D	F	G	A♭	D	G
Forte Name	7-32	5-26	4-22	5-33	5-27	6-33
Hindemith Group	IV	IV	III	II.b.3	III	IV
Hanson symbol		pm³n²s²dt	p²mns²	m⁴s⁴t²	p̄³m²n²s²d	

Sonority	D - E - F - G - A - B		C - D - E - G - A	
Root			(C) G	
Forte Name	6-33		5-35	
Hindemith Group	IV		III	
Hanson symbol	$p^4m^2n^3s^4dt$		$p^4mn^2s^3$	

Sonority	C - D - E - G - A			
Root	(C) G		C	
Forte Name	5-35			
Hindemith Group	III			
Hanson symbol	$p^4mn^2s^3$			

Sonority	C - D - E - G - A		
Root	(C) G		C	
Forte Name	5-35		
Hindemith Group	III		
Hanson symbol	p⁴mn²s³		